2501. Points $O:(0,0)$, $A:(1,0)$, $B:(1,1)$ and $C:(0,1)$ are defined. Two points are then chosen at random from the four.

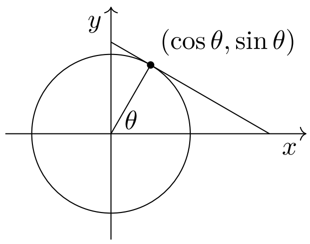
(a) Find the probability that the distance between the two points is 1.

(b) Given that the distance between the two points is $\sqrt{2}$, find the probability that the origin was chosen.

2502. Show that the tangent to $y = e^{-\frac{1}{2}x^2}$ at $x = -1$ crosses the x axis at $x = -2$.

2503. Evaluate $\lim\limits_{k \to 0} \dfrac{(6+k)^3 - 6^3}{(6+k)^2 - 6^2}$.

2504. A unit circle is depicted below:



Show that the tangent to the circle at angle θ crosses the axes at $\operatorname{cosec}\theta$ and $\sec\theta$.

2505. State, with a reason, whether the curve $y = e^x$ intersects the following curves:

(a) $y = e^{-x}$,

(b) $y = -e^x$,

(c) $y = -e^{-x}$.

2506. Write the following in simplified interval notation:

$$\{x \in \mathbb{R} : x^2 - x < 6\} \cap \{x \in \mathbb{R} : x^2 + x < 6\}.$$

2507. Find, in simplified exact form, the coordinates of any stationary points of the following curve:

$$y = \frac{(\ln x)^2}{2 - \ln x}.$$

2508. In this question, the usual convention is followed, in which the vertices of a polygon are named in order around the perimeter.

The convex quadrilateral $ABCD$ has its vertices at $A:(-2,1)$, $B:(2,0)$, $C:(0,3)$ and $D:(x,y)$. Show that $3x + 2y < 6$ and $x + 4y > 2$, and find a third inequality defining the location of D.

2509. Variables x and y have constant rates of change

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b.$$

Find the rate of change of x^2y^2 in terms of x, y, a and b, giving your answer in factorised form.

2510. Prove that, if $\dfrac{d}{dx}(k^x) = k^x$ for $k \neq 0$, then

$$\lim_{h \to 0} \frac{k^h - 1}{h} = 1.$$

2511. The temperature T°C of an oscillating industrial process is modelled as $T = 7\sin t + 24\cos t + 10$.

(a) Find the range of temperatures attained in the oscillation.

(b) When the temperature is above 22.5°C, the system is classified as "warm". Determine the probability, if the system is randomly tested, of the process being classified as warm.

2512. Show that $\dfrac{d}{d\theta}(\sec\theta \operatorname{cosec}\theta) = \sec^2\theta - \operatorname{cosec}^2\theta$.

2513. A set of $2n$ fair coins is tossed. Show that the probability of obtaining more heads than tails is

$$\mathbb{P}(\text{more heads than tails}) = \tfrac{1}{2}\left(1 - \frac{{}^{2n}\mathrm{C}_n}{2^{2n}}\right).$$

2514. A differential equation is given as

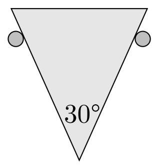
$$\left(\frac{dy}{dx}\right)^2 - y = 0.$$

A quadratic solution curve $y = ax^2 + bx + c$, with $a \neq 0$, is proposed.

(a) Show that $a = \frac{1}{4}$.

(b) Determine the equation of the solution curve which passes through $(2,0)$ and $(-2,4)$.

2515. A cone of mass 2 kg and semi-vertical angle 15° is held in equilibrium by two symmetrical, smooth, cylindrical supports, as represented below.



(a) Find the reaction forces exerted on the cone.

(b) One of the supports breaks, and the cone falls. Explain how the acceleration downwards will compare to g.

2516. A region R is defined as satisfying both $x^2 - 2x < 0$ and $0 < x + y < 1$. Show that R is a parallelogram with area 2.

2517. An integral equation is given as

$$\int \frac{1}{x}\,dx + \int \frac{1}{y}\,dy = 0.$$

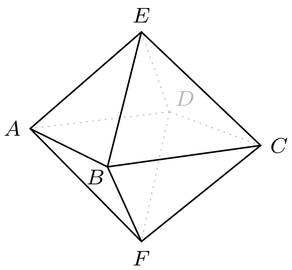
Show that its solution curves are reciprocal graphs.

2518. Sketch $y = 5\sin^2 x + \cos 2x$ for $x \in [-\pi, \pi]$.

2519. Five data, modelled with variables $Z_1, Z_2, ..., Z_5$, are sampled from a large population distributed as $Z \sim N(0,1)$. The mean of these is designated $\bar{Z}$. Find the following probabilities:

(a) $\mathbb{P}(|Z_1| < 1)$,
(b) $\mathbb{P}(|\bar{Z}| < 1)$,
(c) $\mathbb{P}(Z_1 < \bar{Z})$.

2520. A regular unit octahedron is shown below. An ant is walking the surface of the octahedron, starting at A. It chooses an edge at random and walks its length. Upon reaching another vertex, it chooses a new edge at random, never leaving the edges and never travelling the same edge twice.



Find the probability that, when it has walked a distance of three units, the ant is back at point A.

2521. Two parametric vector lines are defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-4t \\ 2-2t \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3+t \\ -3-t \end{pmatrix}.$$

Show that these lines meet in the elliptical region defined by $(x-1)^2 + 2y^2 < 7$.

2522. Express $x^6 - 6x^4$ in polynomial terms of $(x-1)$.

2523. A projectile is launched from a height h above the ground, with fixed initial speed u. Prove that the landing speed will be the same whatever the initial angle of projection.

2524. Show that the parabola $y = 3x^2 + x$ cannot not be transformed into $y = -2x^2 + x + 2$ by any sequence of reflections and rotations.

2525. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

① $x \in \mathbb{R} \setminus \mathbb{Z}$,
② $x \in \mathbb{R} \setminus \mathbb{Q}$.

2526. Prove, from first principles, that

$$\frac{d}{dx}\left(x^{-\frac{1}{2}}\right) = -\tfrac{1}{2}x^{-\frac{3}{2}}.$$

2527. A graph has equation $\mathrm{f}(x) + \mathrm{g}(y) = 0$ for some functions f and g. Write down the equation of the transformed graph, when it is reflected in

(a) the x axis,
(b) the line $y = x$.

2528. Determine the exact value of $\tan \frac{3\pi}{8}$, given that

$$\operatorname{cosec} \tfrac{3\pi}{8} = \sqrt{4 - 2\sqrt{2}}.$$

2529. A collection of k sets of data are to be combined. For $i = 1, 2, ..., k$, they contain n_i data and have mean $\bar{x}_i$. Find a formula, in sigma notation, for $\bar{x}_{\text{total}}$, the mean of the combined set.

2530. Show that the graphs $x^2 - 6x + y^2 + 5 = 0$ and $x^2 + 4x + y^2 + 4y - 3 = 0$ do not intersect, but do approach each other closely.

2531. Prove that, for all positive real numbers x, y, k, a, b,

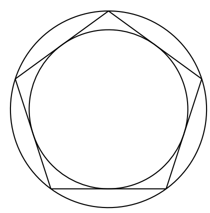
$$x + y > k \implies ax + by > k\min(a, b).$$

2532. A jet engine with a mass of 200 kg is mounted on a smooth track in a laboratory. With the brakes engaged, the engine is started, until it is producing exhaust at a constant rate of 5 kg per second, at a speed of 400 ms^{-1}. Find the initial acceleration of the engine when the brakes are released.

2533. Either prove or provide a counterexample to the following statement:

"If events A and B are independent, and events B and C are independent, then events A and C must be independent."

2534. A regular n-gon has two circles drawn touching it, one inscribed and one circumscribed.



(a) Prove that the ratio of the circumferences of the circles is $1 : \cos \frac{180°}{n}$.
(b) Show that this ratio approaches 1 as $n \to \infty$.

2535. Make x the subject of $qxyz + px^2 + ry^2z^2 = 0$.

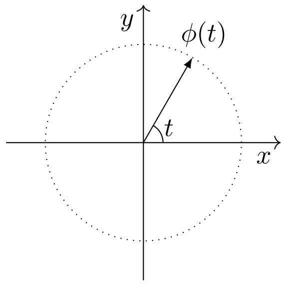
2536. Show that, if $z^2 = 4x + 1$, then $\displaystyle\int \frac{2}{z}\,dx = z + c$.

2537. A fair six-sided die is rolled twice, and the second score is observed to be higher than the first. Find the probability that the second score was a six.

2538. A *phase function* ϕ is defined as

$$\phi(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}.$$

Its derivative is notated $\dot{\phi}(t)$.



(a) Show that $\phi(t)$ and $\dot{\phi}(t)$ are perpendicular unit vectors.

(b) Find $\ddot{\phi}(t)$.

(c) Explain how your answer to (b) proves that, in circular motion at constant speed, acceleration is towards the centre.

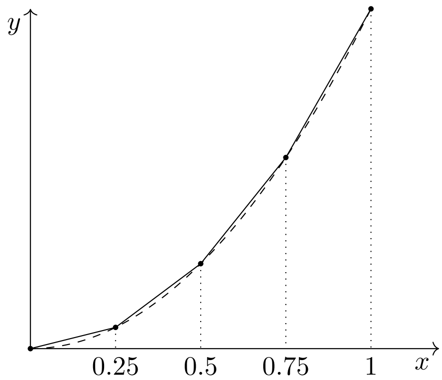
2539. True or false?

(a) $y = x^6 + x^3$ has a local minimum,

(b) $y = x^5 + x^3$ has a local minimum,

(c) $y = x^4 + x^3$ has a local minimum.

2540. The diagram shows the parabola $y = x^2$, with the curve approximated by four straight line segments.



The exact arc length of the parabola (dashed), from $x = 0$ to $x = 1$, is

$$l = \tfrac{1}{4}\left(2\sqrt{5} + \ln\left(2 + \sqrt{5}\right)\right).$$

(a) Find the percentage error, relative to the above value, if four straight line segments are used to approximate the curve.

(b) Explain why such a calculation, regardless of the number of segments, will always give an underestimate of the true arc length.

2541. Find all intersections of the curves

$$\begin{aligned} \log_2 x + 2\log_4 y &= 3, \\ x + y &= 6. \end{aligned}$$

2542. Show that the tangent to $y = x^2$ at point $\left(a, a^2\right)$ makes an angle $\theta = \arctan 2a$ with the x axis.

2543. State, with a reason, whether the following hold:

(a) $\displaystyle\sum_{r=1}^{n} u_r^{-1} \equiv \left(\sum_{r=1}^{n} u_r\right)^{-1}$

(b) $\displaystyle\sum_{r=1}^{n} au_r + bv_r \equiv a\sum_{r=1}^{n} u_r + b\sum_{r=1}^{n} v_r$

(c) $\displaystyle\sum_{r=1}^{n} u_r v_r \equiv \sum_{r=1}^{n} u_r \sum_{r=1}^{n} v_r.$

2544. Curve A has equation $y = x^{\frac{3}{4}} - 2x^{\frac{1}{2}} + x^{\frac{1}{4}}$.

(a) Find the equation of the tangent at O.

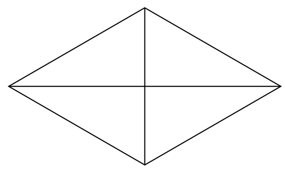
(b) Find and classify any stationary points.

(c) Hence, sketch the curve.

2545. In this question, do not use a calculator.

Solve $x^2 + \dfrac{x-1}{x+1} = 3 - 2x$.

2546. A rhombus has diagonals of length a and b.



Prove that the largest circle which can be inscribed in this rhombus has radius given by

$$r = \frac{ab}{2\sqrt{a^2 + b^2}}.$$

2547. A function is defined, over a suitable domain, by

$$\mathrm{f} : x \mapsto \frac{x+2}{2x+1}.$$

Find a simplified expression for $\mathrm{f}^2(x)$.

2548. For each of the following graphs, given in radians, find the coordinates of any points at which the tangent is parallel to the y axis:

(a) $y = \arcsin x$,

(b) $y = \arccos x$,

(c) $y = \arctan x$.

2549. Show that the locus of $3x^2 + 7xy + 2y^2 = 0$ is a pair of intersecting lines.

2550. Four couples sit down at random at a round table. The probability that everyone ends up sitting next to their partner is denoted p. Show that $p = \frac{2}{105}$.

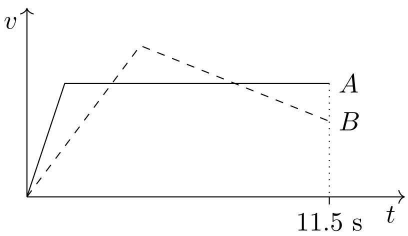
2551. State, with a reason, whether the following gives a well-defined function:

$$\mathrm{h} : \begin{cases} (0,1) \mapsto \mathbb{R} \\ x \mapsto \dfrac{1}{x^5 - x^4}. \end{cases}$$

2552. Find constants A, B, C such that the following is an identity:

$$\frac{1}{x^4 - x^2} \equiv \frac{A}{x^2} + \frac{B}{x+1} + \frac{C}{x-1}.$$

2553. The performances of two sprinters in a 100 metre running race are modelled as follows. A attained a maximum speed of 9 ms^{-1}, and sustained it for the rest of the race; B attained a maximum speed of 11 ms^{-1}, but then decelerated constantly. When A finished after 11.5 seconds, B was 4 m behind, running at 7 ms^{-1}.



(a) Determine A's acceleration at the start.

(b) Find the time at which B hit maximum speed.

2554. The curve $2\ln x - \ln y = k$, where k is a constant, makes up half of the parabola $y = x^2$.

(a) Find the value of k.

(b) Determine, in a similar form, the equation of the other half of $y = x^2$.

2555. A student is performing a binomial hypothesis test, analysing potential bias, either for or against sixes, in a die, and has set up hypotheses as follows:

$$H_0 : p > \tfrac{1}{6},$$
$$H_1 : p < \tfrac{1}{6}.$$

Explain why such hypotheses cannot be used for a test, and provide corrected hypotheses.

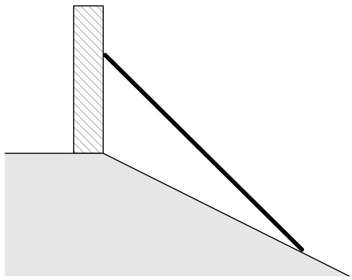
2556. An *Euler brick* is a cuboid whose edges and face diagonals all have integer length. In the smallest such, the face diagonals have lengths 125, 244 and 267. By solving simultaneous equations, find the edge lengths of this Euler brick.

2557. Find the equation of the tangent to the curve $x^2y^3 - 2 = xy^{\frac{3}{2}}$ at $(-1, 1)$, giving your answer in the form $ay = bx + c$, for $a, b, c \in \mathbb{N}$.

2558. Eliminate t from the following equations, to find a relation of the form $\mathrm{f}(x) + \mathrm{g}(y) = 0$:

$$x = 2\cos t,$$
$$y = 5 + 3\sin t.$$

2559. A uniform plank of mass m is leaning against and at 45° to a smooth vertical wall. The plank stands on a rough slope, which runs down away from the wall, of inclination 30°.



(a) By taking moments, show that the reaction at the wall has magnitude $\frac{1}{2}mg$.

(b) By resolving horizontally and vertically, show that, if the plank is on the point of slipping, then the reaction R at the foot of the plank and the coefficient of friction μ must satisfy

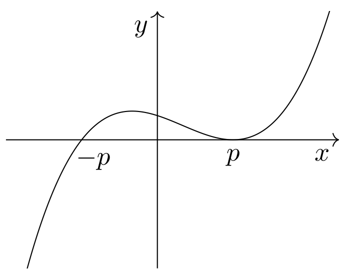
$$R(\sqrt{3} + \mu) = 2mg,$$
$$R(\sqrt{3}\mu - 1) = mg.$$

(c) Hence, show that, for equilibrium,

$$\mu \geq \frac{2 + \sqrt{3}}{2\sqrt{3} - 1}.$$

2560. "The x axis is tangent to the curve $y^2 = x^4 - x^3$." True or false?

2561. A cubic $y = ax^3 + bx^2 + cx + d$ is shown below, with x intercepts at $x = \pm p$:



State, with a reason, whether the following facts are necessarily true:

(a) "a is positive",

(b) "b is negative",

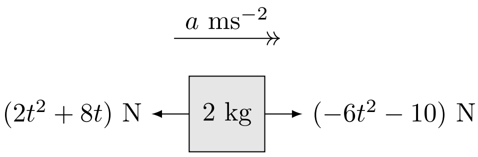
(c) "c is negative",

(d) "d is positive".

2562. The iteration $u_{n+1} = \sqrt{u_n^2 + k}$, for some $k \in \mathbb{R}$, is applied, with first term $u_1 = a$. Find an ordinal formula for u_{n+1} in terms of n.

2563. Prove that $\dfrac{1}{1+\tan x} + \dfrac{1}{1+\cot x} \equiv 1$.

2564. Two variable forces are applied to an object. The forces are detailed in the following diagram:



(a) Find a in terms of t.
(b) Determine the least value of the magnitude of the acceleration in the period $t \in [0, 2]$.

2565. Write down the equation of the reflection of each of the following graphs in the line $y = k$:

(a) $y = ax + k$,
(b) $y = ax^2 + bx + k$.

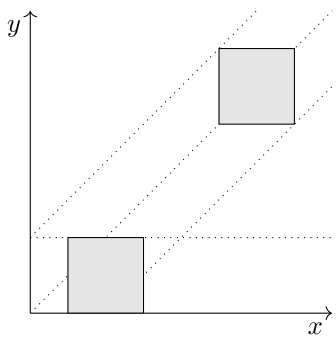
2566. A sample has $\sum x = \sum x^2$. By considering the formula for S_{xx}, show that $\bar{x} \in [0, 1]$.

2567. Prove that a right-angled trapezium which is cyclic must be a rectangle.

2568. If $\dfrac{d^2}{dx^2}(x + y) = 2$, find y in terms of x.

2569. Two squares of unit side length are moving in the (x, y) plane. Their sides are parallel to the axes, and their lower left-hand vertices are at points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, where

$$\mathbf{r}_1 = \begin{pmatrix} t \\ t \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 3-t \\ 0 \end{pmatrix}.$$



Determine whether there are any times at which the squares share a vertex.

2570. Two populations are well modelled by two normal distributions $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$. Explain whether the combined population is well modelled by a normal distribution.

2571. Three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ have the following properties:

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$$
$$\mathbf{a} - \mathbf{b} + 2\mathbf{c} = 0$$

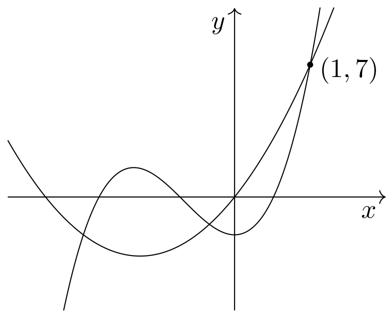
Show that all three vectors are parallel.

2572. The equation $x^4 + ax^2 + bx^2 + ab = 0$ is given, in which the constants a, b are such that $a > b > 0$.

(a) Considering the equation as a quadratic in x^2, show that $\Delta = (a - b)^2$.
(b) Show that the equation has no real roots.

2573. In this question, do not use a calculator.

The diagram shows a cubic and a quadratic, which intersect at $(1, 7)$:



The equations of the curves are

$$y = 3x^3 + 6x^2 - 2,$$
$$y = 2x^2 + 5x.$$

Determine the coordinates of all intersections.

2574. Two particles have positions given by

$$\mathbf{r}_1 = t^3 - t - 5,$$
$$\mathbf{r}_2 = t^2 + 9.$$

Find the time(s) at which their

(a) velocities are the same,
(b) speeds are the same.

2575. Show that $\displaystyle\int_{-1}^{1} \frac{x}{x^2 - 4}\, dx = 0$.

2576. A set S is given as follows, with the sine function defined in radians:

$$S = \left\{ x \in \mathbb{Z} : \sin x < \tfrac{1}{2} \right\}.$$

Prove that S has infinitely many elements.

2577. It is given that the parabola $y = px^2 + qx + r$ has roots $x = a, b$, and is stationary at (u, v). Write down the equation of the parabola with roots $x = a, b$, which is stationary at $(u, -v)$.

2578. On any given day in summer, the probability that I wear a jacket is 10%, and the probability that it rains at some stage is 20%. On days when it rains, the probability that I wear a jacket is 25%.

(a) Find the probability that it doesn't rain and I don't wear a jacket.

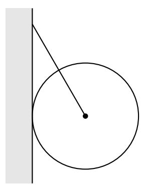
(b) Given that I wear a jacket, find the probability that it rains at some stage.

2579. A logo is designed as the region enclosed by the curve $x = y^2$ and the lines $y = \pm\frac{1}{2}(x-1) \pm 1$.

(a) Sketch the logo, labelling any intercepts.

(b) Find the area of the logo.

2580. A smooth uniform cylinder of radius r and mass m is hung horizontally against a wall by two light, inextensible strings of length $2r$, one at each end, each attached at the axis of symmetry.



Determine the magnitude of

(a) the tension in each string,

(b) the reaction force exerted on the wall,

(c) the total force exerted on the wall.

2581. An iteration is given by $u_n = u_{n-1}^2 - 1$, with first term x. This iteration is periodic, with period 2.

(a) Show that $x^4 - 2x^2 = x$.

(b) Without a calculator, determine all possible values of x.

2582. Simultaneous equations are given as

$$P \operatorname{cosec}\theta = 17,$$
$$P \cot\theta = 8.$$

Find all possible values of P.

2583. A casino inspector suspects that dice at a table are biased against sixes. The inspector observes fifty consecutive rolls, and sees five sixes.

(a) Give suitable hypotheses with which to test the inspector's suspicion.

(b) Conduct a test at the 5% level of significance, giving your conclusion in context.

(c) Explain why, for the test to be meaningful, the fifty observations should all be taken **after** the point the inspector becomes suspicious.

2584. A quadrilateral $PQRS$ has sides with the following lengths: $|PQ| = 5$, $|QR| = 3$, $|RS| = 2$, $|SP| = 1$. Show that $\angle PQR \leq \arccos\frac{5}{6}$.

2585. Region R is defined by simultaneous inequalities given, in terms of constants $a > 1$ and $b > 0$, as

$$y < x^2,$$
$$y > ax^2 + bx.$$

Show that the area of R is $\dfrac{b^3}{6(1-a)^2}$.

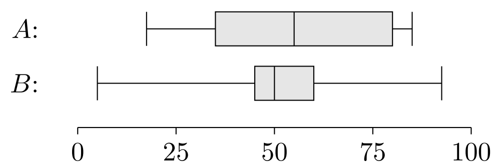
2586. Prove that, if a cubic graph $y = ax^3 + bx^2 + cx + d$ has two stationary points, then these lie the same x distance from the point of inflection.

2587. Solve the simultaneous equations

$$a^2b - 9a\sqrt{b} + 8 = 0,$$
$$a^2 - ab = 0.$$

2588. Show that the shortest distance between the curves $2y = x^2 + 2$ and $2x = y^2 + 2$ is $\sqrt{2}/2$.

2589. The marks for a test taken by two sets of students are summarised in the following diagram:



(a) Explain why neither the range nor the IQR is likely to be useful, on its own, for comparing the spread of sets A and B.

(b) For each set, give the quartiles, e.g. 50-75%, between which the mode is likeliest to lie.

2590. Disprove the following statement:

$$\mathrm{fg}(x) \equiv \mathrm{gf}(x) \implies \mathrm{f}(x) \equiv \mathrm{g}(x).$$

2591. True or false?

(a) For all $x \in \mathbb{R}$, $\sin x > \cos x - 1$.

(b) For all $x \in \mathbb{R}$, $\cos x > \tan x - 1$.

(c) For all $x \in \mathbb{R}$, $\tan x > \sin x - 1$.

2592. Two graphs are defined, for some constant k, by

$$y = kx,$$
$$\sqrt{x-y} + \sqrt{x+y} = 1.$$

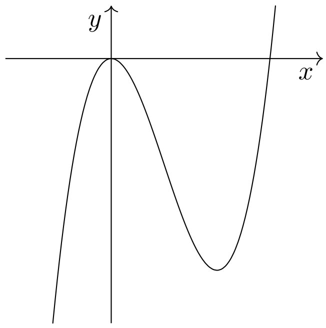
It is given that the graphs intersect at $x = a$.

(a) Show that $\sqrt{a} = \dfrac{1}{\sqrt{1-k} + \sqrt{1+k}}$.

(b) Hence, show that $|k| \leq 1$.

2593. Solve $\log_2\left(2e^x\right) + \log_2\left(e^{2x}\right) = 3$ exactly.

2594. The diagram shows the curve $y = x^3 - 3x^2$. It has rotational symmetry order 2.



Find the coordinates of the centre of rotation.

2595. You are given that the variables x, y, z satisfy

$$\frac{dy}{dx} - \frac{dz}{dx} = 0.$$

State, with a reason, whether any of the variables is necessarily

(a) directly proportional to any other,

(b) linear in any other.

2596. Solve $\sin^3 x + 3\sqrt{3}\cos^3 x = 0$ for $x \in [0, 2\pi]$.

2597. Variables x and y are related as

$$27y = 5e^{3x} - 5e^{-3x} - 3x.$$

Verify that $\dfrac{d^2y}{dx^2} - 9y = x$.

2598. A mechanics student speaks as follows:

"Forces internal to a system only sum to zero if the constituent parts of the system are motionless relative to one another."

State, with a reason, whether this is correct.

2599. In 3D (x, y, z) space, a region R is defined by

$$(x-2)^2 + (y+1)^2 + (z-4)^2 \leq 9.$$

Find the exact volume of region R.

2600. The following notation signifies the rate of change of $\cos x$ with respect to $\sin x$:

$$\frac{d(\cos x)}{d(\sin x)}.$$

(a) Defining $u = \sin x$, find $\dfrac{dx}{du}$ in terms of x.

(b) Explain why $\dfrac{d(\cos x)}{du} = -\sin x \dfrac{dx}{du}$.

(c) Hence, show that $\dfrac{d(\cos x)}{d(\sin x)} = -\tan x$.

——— End of 26th Hundred ———